Comments on “The Glassy Potts Model”

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We report the equivalence of the “Glassy Potts model”, recently introduced by Marinari *et al.* and the “Chiral Potts model” investigated by Nishimori and Stephen. Both models do not exhibit any spontaneous magnetization at low temperature, differently from the ordinary glass Potts model. The phase transition of the glassy Potts model is easily interpreted as the spin glass transition of the ordinary random Potts model.

In a very recent paper Marinari *et al.* [1] introduced a q-component Potts model possessing the same gauge invariance of the Ising spin glass, thus inhibiting the presence of spontaneous magnetization at low temperature. Their model turns out to be a good candidate to describe the physical properties of the real glasses whose glassy phase must extend from T_c down to $T = 0$. In their Hamiltonian

$$H = -\sum_{i,j} \delta_{\sigma_i;\Pi_{ij}(\sigma_j)} \tag{1}$$

the spin variables take the values $\sigma_i = 0, 1, \ldots, q-1$, and the role of the quenched disorder is played by the random quenched *state permutations* Π_{ij}. Thus a non zero contribution to the Hamiltonian is given when $\sigma_i = \Pi_{ij}(\sigma_j)$, whereas in the standard Potts model (see [4] and references in [1]) this is the case when the two spins are found in the same state $\sigma_i = \sigma_j$. Actually a gauge-invariant Hamiltonian (chiral Potts model) has been introduced many years ago by Nishimori and Stephen [2], although in a different form as generalization of Ising spin glass to the Potts glass

$$H = -\sum_{ij} \frac{1}{q} \sum_{r=0}^{q-1} J_{ij}^{(r)} \sigma_i^r \sigma_j^{q-r} \tag{2}$$

where the spins are written in the complex representation (each spin has two components and $(\sigma_i^r)^* = \sigma_i^{q-r}$), each state being represented by one of the q roots of unity, and J_{ij} are complex random quenched variables. The additional condition on the coupling constants $\left(J_{ij}^{(r)}\right)^* = J_{ij}^{(q-r)}$ insures the realness of the expression (2). Of course, when $q = 2$, the Hamiltonian (2) reduces to the well know Ising spin glass problem. Moreover it is worthy to be noticed that when the $\{J\}$'s are constant, one recovers the usual non random Potts model ($H = -J\sum_{ij} \delta_{\sigma_i;\sigma_j}$), because of the following formula

$$\delta_{\sigma_i;\sigma_j} = \frac{1}{q} \sum_{r=0}^{q-1} \sigma_i^r \sigma_j^{q-r} \tag{3}$$

If one considers a discrete distribution of the coupling constant

$$J_{ij}^{(r)} = \tau_{ij}^r, \tag{4}$$

$\{\tau\}$ being a root of the unity associated to the link $\langle ij\rangle$with some probability weight, it is straightforward to check that one recovers the Hamiltonian (1), since the Hamiltonian differs from zero when $\sigma_i = \tau_{ij}\sigma_j$, τ_{ij} acting as a random permutation of the spin values. In the mean field limit, the chiral Potts model can be easily solved by means of the replica trick [3], thus leading at the minimization of the following free energy density

$$\begin{aligned}
-\beta f = & -\frac{J_0\beta}{2q} \sum_{\alpha} \sum_{r=1}^{q-1} \left[(m_{1;r}^{\alpha})^2 + (m_{2;r}^{\alpha})^2\right] - \\
& -\frac{1}{2}\left(\frac{J\beta}{q}\right)^2 \sum_{(\alpha\beta)} \sum_{r=1}^{q-1} \left[(Q_{1;r}^{(\alpha\beta)})^2 + (Q_{2;r}^{(\alpha\beta)})^2\right] + \\
& + \log \mathrm{Tr} \exp \left\{ \left(\frac{J\beta}{q}\right)^2 \sum_{(\alpha\beta)} \sum_{r=1}^{q-1} \left[Q_{1;r}^{(\alpha\beta)} \mathrm{Re}\left[(\sigma^\alpha)^r (\sigma^\beta)^{q-r}\right] + \right.\right. \\
& \left. + Q_{2;r}^{(\alpha\beta)} \mathrm{Im}\left[(\sigma^\alpha)^r (\sigma^\beta)^{q-r}\right]\right] + \\
& \left. + \left(\frac{J_0\beta}{q}\right) \sum_{\alpha} \sum_{r=1}^{q-1} \left[m_{1;r}^{\alpha} \mathrm{Re}\left[(\sigma^\alpha)^r\right] + m_{2;r}^{\alpha} \mathrm{Im}\left[(\sigma^\alpha)^r\right]\right] \right\}
\end{aligned} \tag{5}$$

It is straightforward to check that for $q = 2$, the imaginary parts in (5) vanishes and one obtains the replicated SK free energy. Here the coefficient of the magnetization is simply $J_0\beta/q$ to be compared with the coefficient one gets in the standard Potts model [4]

$\beta\left(J_0+\frac{1}{2}(q-2)\beta J^2\right)$, responsible for the ferromagnetic phase at low temperature in absence of magnetic field. In order to investigate the para-glass (isotropic) transition, where the magnetization vanishes $m^\alpha_{1;r}=m^\alpha_{2;r}=0$, we assume that the spin glass order parameters do not depend on r. This implies $Q^{(\alpha\beta)}_{1;r}=Q^{(\alpha\beta)}$ and $Q^{(\alpha\beta)}_{2;r}=0$. Therefore, from (3) and (5) the free-energy to be minimized simply reads

$$-\beta f \;=\; -\frac{1}{2}(q-1)\left(\frac{J\beta}{q}\right)^2\sum_{(\alpha\beta)}\left(Q^{(\alpha\beta)}\right)^2 \;+$$

$$+\;\log\mathrm{Tr}\exp\left[\left(\frac{J\beta}{q}\right)^2\sum_{(\alpha\beta)}Q^{(\alpha\beta)}\left(\delta_{\sigma^\alpha\sigma^\beta}-\frac{1}{q}\right)\right] \qquad (6)$$

which turns out to be exactly the same free energy of the standard random Potts model [4], with the practical advantage that the glass phase extends down to $T=0$. Therefore for $q=4$, the transition from the paramagnetic phase to the glass phase is discontinuous with one step replica symmetry breaking at least above the upper critical dimension. On the other hand, the numerical calculations performed in [1] seem to indicate that the REM-like landscape of the free energy holds also in d=4.

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